# Домашна работа 2 по ДИС 1. Компютърни науки, 1 курс, 2 поток

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20 Ноември 2021

**Задача 1.** Дадена е редицата

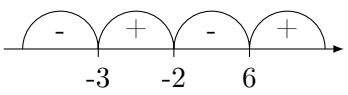
$$a_1 = \lambda,\ a_{n+1} = \frac{2a_n^2 - a_n + 6}{a_n - 6}$$

Да се изследва за сходимост спрямо стойностите на параметъра $\lambda \neq 6$.

**Решение:** Разглеждаме за монотонност:

$$a_{n+1} - a_n = \frac{2a_n^2 - a_n + 6}{a_n - 6} - a_n = \frac{2a_n^2 - a_n + 6 - a_n^2 + 6a_n}{a_n - 6} = \frac{a_n^2 + 5a_n + 6}{a_n - 6} = \frac{(a_n + 3)(a_n + 2)}{a_n - 6}$$

$\implies$ Корените за $a_n$ са -3, -2 и 6



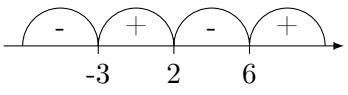

Ако съществува $l = \lim\limits_{n\to\infty} a_n$, то

$$l = \frac{2l^2 - l + 6}{l - 6} \qquad l^2 - 6l = 2l^2 - l + 6 \qquad 0 = l^2 + 5l + 6 \implies l_1 = -2\ ,\ l_2 = -3$$

Разглеждаме положението на $a_{n+1}$ спрямо -3, -2 и 6

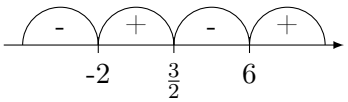
$$a_{n+1} - (-3) = \frac{2a_n^2 - a_n + 6}{a_n - 6} + 3 = \frac{2a_n^2 - a_n + 6 + 3a_n - 18}{a_n - 6} = \frac{2a_n^2 + 2a_n - 12}{a_n - 6}$$

$\implies$ Корените за $a_n$ са -3, 2 и 6



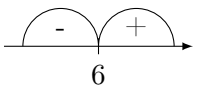

$$a_{n+1} - (-2) = \frac{2a_n^2 - a_n + 6}{a_n - 6} + 2 = \frac{2a_n^2 - a_n + 6 + 2a_n - 12}{a_n - 6} = \frac{2a_n^2 + a_n - 6}{a_n - 6}$$

$\implies$ Корените за $a_n$ са -2, $\frac{3}{2}$ и 6




$$a_{n+1} - 6 = \frac{2a_n^2 - a_n + 6}{a_n - 6} - 6 = \frac{2a_n^2 - a_n + 6 - 6a_n + 36}{a_n - 6} = \frac{2a_n^2 - 7a_n + 42}{a_n - 6} > 0$$

$\implies$ Единствения корен за $a_n$ е 6




Разглеждаме положението на $a_{n+1}$ спрямо $\frac{3}{2}$ и 2

$$a_{n+1} - \frac{3}{2} = \frac{2a_n^2 - a_n + 6}{a_n - 6} - \frac{3}{2} = \frac{2a_n^2 - a_n + 6 - \frac{3}{2}a_n + 9}{a_n - 6} = \frac{2a_n^2 - \frac{5}{2}a_n + 15}{a_n - 6} > 0$$

$\implies$ Единствения корен за $a_n$ е 6

$$a_{n+1}-2=\frac{2a_n^2-a_n+6}{a_n-6}-2=\frac{2a_n^2-a_n+6-2a_n+12}{a_n-6}=\frac{2a_n^2-3a_n+18}{a_n-6}>0$$

$\Longrightarrow$ Единствения корен за $a_n$ е 6

Разглеждаме положението на $a_{n+1}$

$$a_{n+1}=\frac{2a_n^2-a_n+6}{a_n-6}>0 \Longrightarrow \text{Единствения корен за } a_n \text{ е } 6$$

| $a_n$ | $-\infty$ | -3 | | -2 | | $\frac{3}{2}$ | | 2 | | 6 | $+\infty$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $a_{n+1}-a_n$ | - | $\vdots$ | + | $\vdots$ | - | $\vdots$ | - | $\vdots$ | - | $\vdots$ | + |
| $a_{n+1}-(-3)$ | - | $\vdots$ | + | $\vdots$ | + | $\vdots$ | + | $\vdots$ | - | $\vdots$ | + |
| $a_{n+1}-(-2)$ | - | $\vdots$ | - | $\vdots$ | + | $\vdots$ | - | $\vdots$ | - | $\vdots$ | + |
| $a_{n+1}-6$ | - | $\vdots$ | - | $\vdots$ | - | $\vdots$ | - | $\vdots$ | - | $\vdots$ | + |
| $a_{n+1}-\frac{3}{2}$ | - | $\vdots$ | - | $\vdots$ | - | $\vdots$ | - | $\vdots$ | - | $\vdots$ | + |
| $a_{n+1}-2$ | - | $\vdots$ | - | $\vdots$ | - | $\vdots$ | - | $\vdots$ | - | $\vdots$ | + |
| $a_{n+1}$ | - | $\vdots$ | - | $\vdots$ | - | $\vdots$ | - | $\vdots$ | - | $\vdots$ | + |

**I. Случай** $\lambda<-3$ Знаем, че $a_{n+1}-a_n$ е винаги намаляваща $\Longrightarrow \lim\limits_{n\to\infty} a_n=-\infty$

**II. Случай** $\lambda\in(-3,-2)$

**III. Случай** $\lambda\in(-2,\frac{3}{2})$

**IV. Случай** $\lambda\in(\frac{3}{2},2)$

**V. Случай** $\lambda\in(2,6)$ Знаем, че $a_{n+1}-a_n$ е винаги намаляваща $\Longrightarrow \lim\limits_{n\to\infty} a_n=-\infty$

**VI. Случай** $\lambda>6$ Знаем, че $a_{n+1}-a_n$ е винаги растяща $\Longrightarrow \lim\limits_{n\to\infty} a_n=+\infty$

**Задача 2.** Да се пресметне границата

$$\lim_{x\to\infty}\frac{\sqrt{2x^2+1}-\sqrt[3]{4x^2-1}}{\sqrt[4]{x^4+7}-\sqrt[5]{x^4-3}}$$

**Решение:**

$$\lim_{x\to\infty}\frac{\sqrt{2x^2+1}-\sqrt[3]{4x^2-1}}{\sqrt[4]{x^4+7}-\sqrt[5]{x^4-3}}=\lim_{x\to\infty}\frac{\cancel{x}\sqrt{2+\frac{1}{x^2}}-\cancel{x}\sqrt[3]{\frac{4}{x}-\frac{1}{x^3}}}{\cancel{x}\sqrt[4]{1+\frac{7}{x^4}}-\cancel{x}\sqrt[5]{\frac{1}{x}-\frac{3}{x^4}}}=L$$

$$\sqrt{2}\xrightarrow[x\to\infty]{}\sqrt{2}\,,\ \frac{1}{x^2}\xrightarrow[x\to\infty]{}0\,,\ \frac{4}{x}\xrightarrow[x\to\infty]{}0\,,\ \frac{1}{x^3}\xrightarrow[x\to\infty]{}0\,,$$

$$\sqrt[4]{1}\xrightarrow[x\to\infty]{}\sqrt[4]{1}\,,\ \frac{7}{x^4}\xrightarrow[x\to\infty]{}0\,,\ \frac{1}{x}\xrightarrow[x\to\infty]{}0\,,\ \frac{3}{x^4}\xrightarrow[x\to\infty]{}0 \qquad \Longrightarrow L=\frac{\sqrt{2}}{\sqrt[4]{1}}=\sqrt{2}$$

**Задача 3.** Да се пресметне границата

$$\lim_{x\to0}\frac{\sqrt[3]{5x+1}-\sqrt[4]{7x+1}}{\operatorname{arctg}(\ln(x+1))}$$

**Решение:**

$\operatorname{arctg}(\ln(x+1))\sim\ln(x+1)\quad x\to0$ $\qquad \sqrt[3]{5x+1}=\underbrace{(1+5x)^{\frac{1}{3}}-1}_{\sim\frac{5x}{3}\quad x\to0}+1$ $\qquad \sqrt[4]{7x+1}=\underbrace{(1+7x)^{\frac{1}{4}}-1}_{\sim\frac{7x}{x}\quad x\to0}+1$

$$\Longrightarrow \lim_{x\to0}\frac{\sqrt[3]{5x+1}-\sqrt[4]{7x+1}}{\operatorname{arctg}(\ln(x+1))}\overset{(\sim)}{=\!=}\lim_{x\to0}\frac{\frac{5x}{3}+1-\left(\frac{7x}{4}+1\right)}{\ln(x+1)}=\lim_{x\to0}\frac{\frac{20x-21x}{12}}{\ln(x+1)}=\lim_{x\to0}-\frac{x}{12\ln(x+1)}=L$$

$$\frac{x}{\ln(x+1)}\xrightarrow[x\to0]{}1\Longrightarrow L=-\frac{1}{12}$$

**Задача 4.** Да се пресметне $f^{(n)}(x)$, където

$$f(x) = \frac{x^3}{\sqrt{1-7x}}$$

**Решение:**

$$g(x) = \frac{1}{\sqrt{1-7x}} = (1-7x)^{\frac{1}{2}} \qquad h(x) = x^3 \qquad \Longrightarrow f(x) = g(x)\cdot h(x)$$

$$g^{(n)}(x) = \left((1-7x)^{\frac{1}{2}}\right)^{(n)} = \frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-n+1\right)\cdot x^{\frac{1}{2}-n}\cdot 1^n =$$
$$= \frac{1(-1)(-3)\cdots(1-2n+2)}{2^n}\cdot x^{\frac{1}{2}-n}$$

$$h^{(0)}(x) = h(x) = x^3 \quad h'(x) = 3x^2 \quad h''(x) = 6x \quad h'''(x) = 6 \quad h^{IV}(x) = 0 \quad \Longrightarrow h^{(n)}(x) = 0 \quad n \geq 4$$

$$f^{(n)} = \sum_{k=0}^{n} g^{(n-k)}h^{(k)} = \binom{n}{0}g^{(n)}h^{(0)} + \binom{n}{1}g^{(n-1)}h^{(1)} + \binom{n}{2}g^{(n-2)}h^{(2)} + \binom{n}{3}g^{(n-3)}h^{(3)} +$$
$$+ \underbrace{\binom{n}{4}g^{(n-4)}h^{(4)} + \cdots + \binom{n}{0}g^{(0)}h^{(n)}}_{0} =$$
$$= 1\cdot\frac{(-1)(-3)\cdots(3-2n)}{2^n}\,x^{\frac{1}{2}-n}\cdot x^3 + n\cdot\frac{(-1)(-3)\cdots(5-2n)}{2^n}\,x^{\frac{3}{2}-n}\cdot 3x^2 +$$
$$+ \frac{n!}{2(n-2)!}\cdot\frac{(-1)(-3)\cdots(7-2n)}{2^n}\,x^{\frac{5}{2}-n}\cdot 6x + \frac{n!}{6(n-3)!}\cdot\frac{(-1)(-3)\cdots(9-2n)}{2^n}\,x^{\frac{7}{2}-n}\cdot 6 =$$
$$= \frac{(-1)(-3)\cdots(3-2n)}{2^n}\,x^{\frac{7}{2}-n} + 3n\cdot\frac{(-1)(-3)\cdots(5-2n)}{2^n}\,x^{\frac{7}{2}-n} +$$
$$+ 3n(n-1)\cdot\frac{(-1)(-3)\cdots(7-2n)}{2^n}\,x^{\frac{7}{2}-n} + n(n-1)(n-2)\cdot\frac{(-1)(-3)\cdots(9-2n)}{2^n}\,x^{\frac{7}{2}-n} =$$
$$= x^{\frac{7}{2}-n}\,\frac{(-1)(-3)\cdots(9-2n)}{2^n}\left((8-2n)\cdots(3-2n) + 3n\cdot(8-2n)\cdots(5-2n) +\right.$$
$$\left. +3n^2\cdot(8-2n)(7-2n) + n^3\right)$$

**Задача 5.** Да се пресметне границата

$$\lim_{x\to 0}\left(\frac{\sin x}{\arcsin x}\right)^{\frac{1}{x^2}}$$

**Решение:**

$$\lim_{x\to 0}\left(\frac{\sin x}{\arcsin x}\right)^{\frac{1}{x^2}} = \lim_{x\to 0} e^{\ln\left(\frac{\sin x}{\arcsin x}\right)^{\frac{1}{x^2}}} = \lim_{x\to 0} e^{\frac{1}{x^2}\cdot\ln\left(\frac{\sin x}{\arcsin x}\right)} = e^l$$

$$l = \lim_{x\to 0} \frac{1}{x^2} \ln\left(\frac{\sin x}{\arcsin x}\right) \qquad \ln\left(\frac{\sin x}{\arcsin x} - 1 + 1\right) \sim \frac{\sin x}{\arcsin x} - 1 \quad x \to 0$$

$$\Longrightarrow l \overset{(\sim)}{=\!=} \lim_{x\to 0} \frac{1}{x^2}\left(\frac{\sin x}{\arcsin x} - 1\right) = \lim_{x\to 0} \frac{1}{x^2}\left(\frac{\sin x - \arcsin x}{\arcsin x}\right) \qquad \arcsin x \sim x \quad x \to 0$$

$$\Longrightarrow l \overset{(\sim)}{=\!=} \lim_{x\to 0} \frac{1}{x^2}\left(\frac{\sin x - \arcsin x}{x}\right) = \lim_{x\to 0} \frac{\sin x - \arcsin x}{x^3} \overset{\text{Лопитал } \left[\frac{0}{0}\right]}{=\!=\!=\!=} \lim_{x\to 0} \frac{\cos x - \frac{1}{\sqrt{1-x^2}}}{3x^2}$$

$$\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}} \qquad \left((1-x^2)^{-\frac{1}{2}}\right)' = -\frac{1}{2}(1-x^2)^{-\frac{3}{2}} \cdot \left((1-x^2)\right)' = x(1-x^2)^{-\frac{3}{2}}$$

$$\Longrightarrow l \overset{\text{Лопитал } \left[\frac{0}{0}\right]}{=\!=\!=\!=\!=\!=} \lim_{x\to 0} \frac{-\sin x - x(1-x^2)^{-\frac{3}{2}}}{6x}$$

$$\left(x(1-x^2)^{-\frac{3}{2}}\right)' = \left(\frac{x}{(1-x^2)^{\frac{3}{2}}}\right)' = \frac{(1-x^2)^{\frac{3}{2}} + \frac{3}{2}(1-x^2)^{\frac{1}{2}}2x \cdot x}{(1-x^2)^3} = \frac{(1-x^2)\sqrt{1-x^2} + 3x^2\sqrt{1-x^2}}{(1-x^2)^3} =$$

$$= \frac{2x^2+1}{(1-x^2)^{\frac{5}{2}}}$$

$$\Longrightarrow l \overset{\text{Лопитал } \left[\frac{0}{0}\right]}{=\!=\!=\!=\!=\!=} \lim_{x\to 0} \frac{-\cos x - \frac{2x^2+1}{(1-x^2)^{\frac{5}{2}}}}{6} \qquad \cos x \xrightarrow[x\to 0]{} 1 \qquad \frac{2x^2+1}{(1-x^2)^{\frac{5}{2}}} \xrightarrow[x\to 0]{} 1 \qquad \Longrightarrow l = -\frac{1}{3}$$

Следователно, търсената граница е: $e^l = e^{-\frac{1}{3}} = \boxed{\frac{1}{\sqrt[3]{e}}}$